PIANO TUNING THEORY

Chapter 3.6 - Cents
Draft

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3.6 CENTS

A semitone is 100 cents. An octave is 1200 cents. If you do not need or want to know the math works, you can skip to section 3.8 THE PYTHOGOREAN COMMA. Otherwise, you may find how the math was developed interesting.

THE MATH OF CENTS or MAKING SENSE OF CENTS

When we think of intervals, we imagine each identical interval as being the same size. We imagine all octaves, for example, as the same size, no matter where they occur in the piano.

But when we talk about intervals in terms of the frequencies, or Hertz between the notes, it doesn't work out evenly. Look at the number of Hertz between each octave note A in figure 3-5.
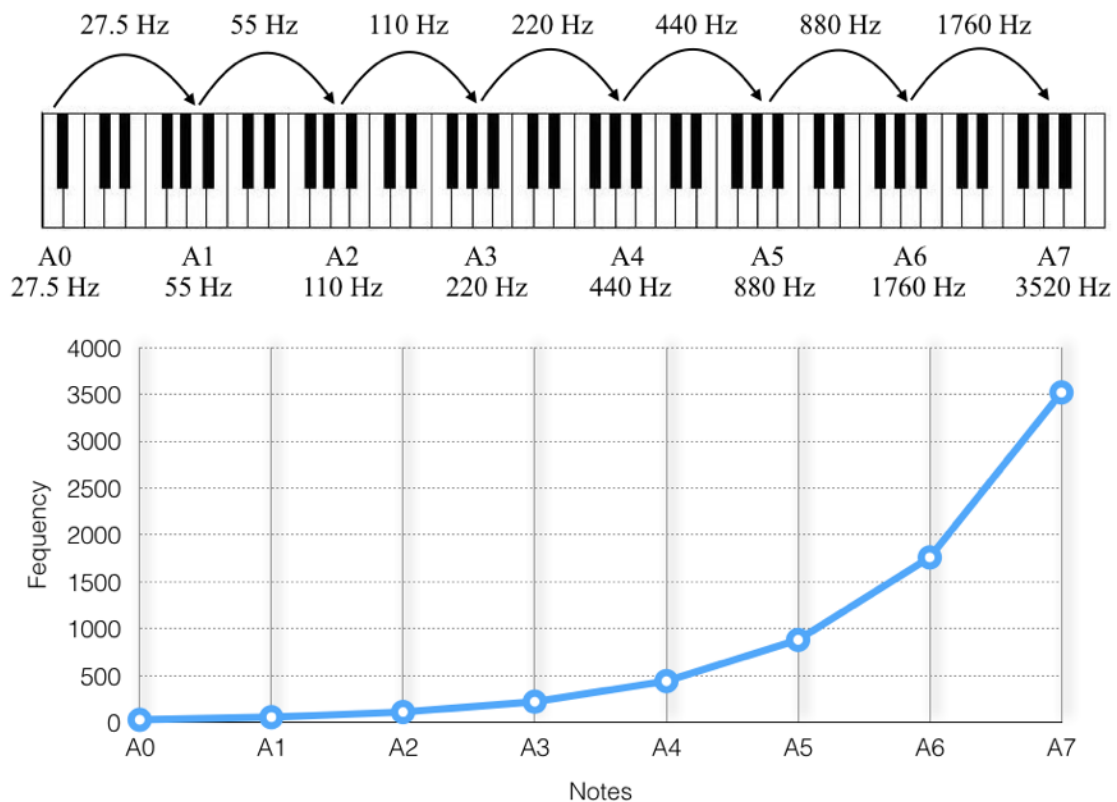

Figure 3-5. Showing How Frequency vs Note
is not a Linear, or Even, Relationship.

From figure 3-5, we can see that, if we want to talk about the size of an interval numerically, and have those numbers be the same for each identical interval, we need a different system, because Hertz doesn't work.

Figure 3-5 shows a non-linear relationship between frequency and note; the line is curved. We need a linear relationship, a straight line, between some unit other than Hertz, and the notes, so we can have all identical intervals measure as the same value.

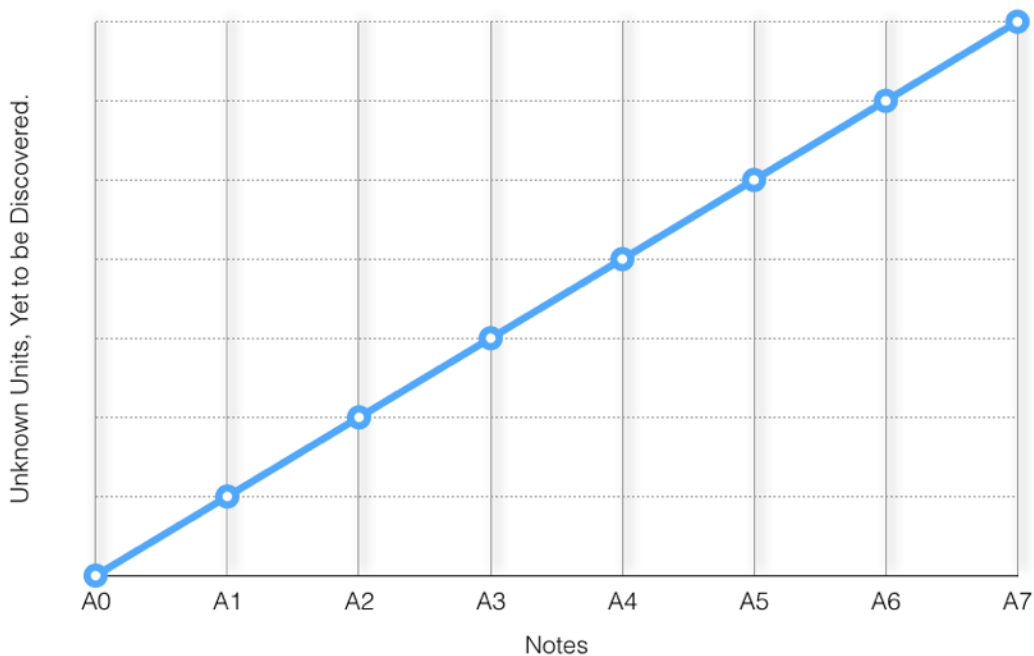We need to find a unit of measuement that produces the straight line relationship like that depicted in figure 3-6.

Figure 3-6. We are looking for a linear relationship.

GETTING THE LINEAR RELATIONSHIP

From figure 3-5, we can see that the octave sizes increase as we go up the piano.

We call this a logarithmic relationship because the difference between each number is not the same. In this case, each frequency is multiplied by 2 to get the next frequency; the frequency doubles every octave.

A logarithmic relationship is one where each step is not the same size. The difference between steps changes by some multiple other than 1.

As has been stated already, we are looking for a *linear* relationship; one where each step difference increases by a multiple of 1. I.e. each step is the same size. That will be our straight line.

The log function can help us figure this out. The next section will be math heavy, but just keep in mind that we need to use the log function to figure this problem out. The next section discusses what the log function is and what it means.

3.6.1 LOGARITHMS

In order to understand what a logarithm is and how it works, we need to understand inverse operations.

OPERATIONS AND INVERSE OPERATIONS

Examples of *Operations* are addition, subtraction, multiplication, division, etc.

An *Inverse Operation* is the opposite of the operation it describes.

Example: Subraction is the inverse operation of addition. Addition is the inverse operation of subtraction. The inverse operation of division is multiplication. And so on.

The logarithmic operation is the inverse of the exponential operation, which I'll discuss soon.

Addition, subtraction, multiplication, and division, are all called *operations,* because they operate on two numbers to produce a third number which is the result.

For example, consider these:

$2 \times 3 = 6$ Multiplication operates on 2 and 3 to produce 6.

$1 + 2 = 3$ Addition operates on 1 and 2 to produce 3.

To illustrate the inverse operation, we can take the two numbers around the = sign, reverse their order, and then use the inverse operation on them. This produces the first number as the result.

Examples:

$$2 \quad \times \quad 3 \quad = \quad 6$$
$$6 \quad \div \quad 3 \quad = \quad 2$$

$$1 \quad + \quad 2 \quad = \quad 3$$
$$3 \quad - \quad 2 \quad = \quad 1$$

That's how inverse operations work.

THE EXPONENTIAL OPERATION

Now, the exponential operation, is basically taking a number and multiplying it by itself a certain number of times.

Example:

2 x 2 x 2 x 2

is 2 multiplied by itself 4 times.

We write it as

$$2^4$$

and we say, "2 to the power 4".

2 is called the *base*, 4 is called the *exponent*, and the answer, 2^4, is called the *power*.

Now, to develop the cents formula, we will need an operation that is the inverse of the exponential operation, For example, we will need an operation that will give us back the exponent as the answer.

We can use logarithms. The logarithm operation is the inverse of the exponential operation.

But first, let's solve the above expression, 2^4, because we need three numbers. Two to operate on, and the answer. So, let's do it,

$$2^4 = 2 \text{ x } 2 \text{ x } 2 \text{ x } 2$$
$$= 2 \text{ x } 2 \text{ x } 4$$
$$= 2 \text{ x } 8$$
$$= 16$$

So,

$$2^4 = 16$$

And we have three numbers: 2, 4, and 16.

The exponential operation can be written as,

$$A^B = C$$

In our example, A = 2, B = 4, and C = 16. i.e.,

$$A^B = C$$

$$2^4 = 16$$

Now, to develop the cents formula, we will need an operation that gives us B, or 4, as the answer.

The logarithm operation does that.

We say, If,

$$A^B = C$$

then

$$\log_A(C) = B$$

Substituting our numbers, we get,

$$\log_2(16) = 4$$

Note that log needs a base, and in our case, 2 is the base.

The above expression is read as, "Log to the base 2 of 16 equals 4".

Again, if,

$$A^B = C$$

then we can say,

$$\log_A(C) = B$$

That's how the logarithmic operation works.

3.6.2 FREQUENCY AND CENTS

Let's look at figure 3-5 again, labelled as figure 3-7 below. It is the chart of the ideal frequencies of all the A's in the piano.
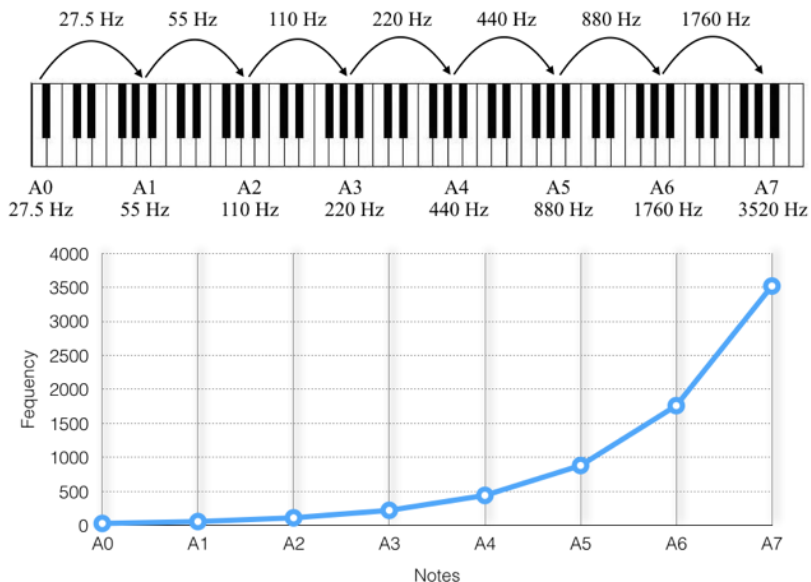

Figure 3-7. Showing How Frequency vs Note
is not a Linear, or Even, Relationship.

Table 3-5 shows the A's and the frequencies of figure 3-7 in table form.

A	Frequency(Hz)
A0	27.5
A1	55
A2	110
A3	220
A4	440
A5	880
A6	1760
A7	3520

Table 3-5. A's and their frequencies.

Let's list each A's octave number and add that to table 3-5 to produce table 3-6.

A	Octave Number	Frequency(Hz)
A0	0	27.5
A1	1	55
A2	2	110
A3	3	220
A4	4	440
A5	5	880
A6	6	1760
A7	7	3520

Table 3-6. Each A, with its octave number and frequency.

Let's add another column that shows what we have to multiply A0 by, in order to get the frequency of each A. Let's call it the 27.5 multiple.

Example: From table 3-7 below, we see that 27.5 x 16 = 440,.

A	Octave Number	27.5 Multiple	Frequency(Hz)
A0	0	1	27.5
A1	1	2	55
A2	2	4	110
A3	3	8	220
A4	4	16	440
A5	5	32	880
A6	6	64	1760
A7	7	128	3520

Table 3-7. A's, octave number, and 27.5 multiple, with frequency.

This next section deals with powers to the base. Understanding powers to the base 2 is another tool that we will need in order to develop the cents formula.

Remember that *power* is just another word for "the answer" when we are talking of exponents.

POWERS TO THE BASE 2

When we use a base of 2 and start solving for a variety of exponents, we get the following powers, or answers.

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$

Notice it is just a doubling pattern. Also notice that the answers are the same as the 27.5 multiples in table 3.7, except we are missing the first multiple.

Well, it turns out that a property of exponents is that, if we take any base, call it x,

$$x^0 = 1$$

So now we have all the powers to the base 2, from 0 to 7, see table 3.8, and they all equal the 27.5 multiples.

X	2^x
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 2 \times 2 = 4$
3	$2^3 = 2 \times 2 \times 2 = 8$
4	$2^4 = 16$
5	$2^5 = 32$
6	$2^6 = 64$
7	$2^7 = 128$

Table 3-8. The Powers of Base 2.

CALCULATING FREQUENCY AND OCTAVE NUMBER

Let's create table 3-9 by adding a column with the base 2 exponents, beside the 27.5 multiples in table 3-7.

A	Octave Number	27.5 Multiple	x where 2^x = 27.5 Multiple	Frequency(Hz)
A0	0	1	0	27.5
A1	1	2	1	55
A2	2	4	2	110
A3	3	8	3	220
A4	4	16	4	440
A5	5	32	5	880
A6	6	64	6	1760
A7	7	128	7	3520

Table 3-9. A's and base 2 exponents
that produce the 27.5 multiples.

Notice in table 3-9, that the base 2 exponents are the same numbers as the octave number. That will be useful later.

Now we can write a formula that can give us the frequency of any A, if we know the octave number, using

$$F_n = 27.5 \times 2^n \qquad\qquad (1)$$

Where,

n is the octave number and,
F_n is the frequency of the nth octave A.

Example,

What is the frequency of A4?

Solution;

The frequency of A4 is given by (1),

$$F_n = 27.5 \times 2^n$$

where n = 4.

$$F_n = 27.5 \times 2^n$$
$$F_4 = 27.5 \times 2^4$$
$$F_4 = 27.5 \times 16$$
$$F_4 = 440$$

So, we can see that our formula works nicely.

Now, what about if we were given the frequency and we wanted to find the octave number?

Let's start with (1),

$$F_n = 27.5 \times 2^n \qquad\qquad (1)$$

and solve for n. the octave number.

Here's how we do it:

Flip the relationship, (1), around to put the n expression on the left of the equals sign,

$$27.5 \times 2^n = F_n \qquad\qquad (2)$$

Divide both sides of (2) by 27.5

$$2^n = F_n / 27.5 \qquad\qquad (3)$$

Now, let's take the log of base 2 of both sides of (3).

$$\log_2(2^n) = \log_2(F_n/27.5) \qquad (4)$$

$\log_2(2^n)$ just equals n because log (to the base 2) and exponent (to the base 2) are inverse operations.

So the formula becomes,

$$n = \log_2(F_n/27.5) \qquad (5)$$

Notice we are taking the log of a fraction.

If you take the log of a fraction, the answer is the same as when you take the log of the numerator (top) minus the log of the denominator (bottom).

I.e. if we have,
$$\log_x(a/b), \qquad (5.1)$$

then it is also equal to,

$$\log_x(a) - \log_x(b) \qquad (5.2)$$

Think: same base, but subtract (top - bottom)

So,
$$n = \log_2(F_n/27.5) \qquad (5)$$

can be written as,

$$n = \log_2(F_n) - \log_2(27.5) \qquad (6)$$

See? Same base, and top minus bottom.

Since $\log_2(27.5)$ is computable, we get,

$$n = \log_2(F_n) - 4.781359714 \qquad (7)$$

(7) is our formula for calculating the octave number given the frequency.

Example: In what octave does A = 440Hz?

Solution,

Use formula (7),

$$n = \log_2(F_n) - 4.781359714 \quad (7)$$

and substitie $F_n = 440$.

$$n = \log_2(F_n) - 4.781359714$$
$$n = \log_2(440) - 4.781359714$$
$$n = 8.781359714 - 4.781359714$$
$$n = 4$$

So, the A that has the pitch of 440Hz, occurs at A4. Nice.

3.6.3 LET'S TALK CENTS

Let's look at the relationship between octave number (or base 2 exponent of 27.5 multiple) vs note.

Figure 3-8 shows a graph of base 2 exponents vs note. Notice that it is LINEAR!
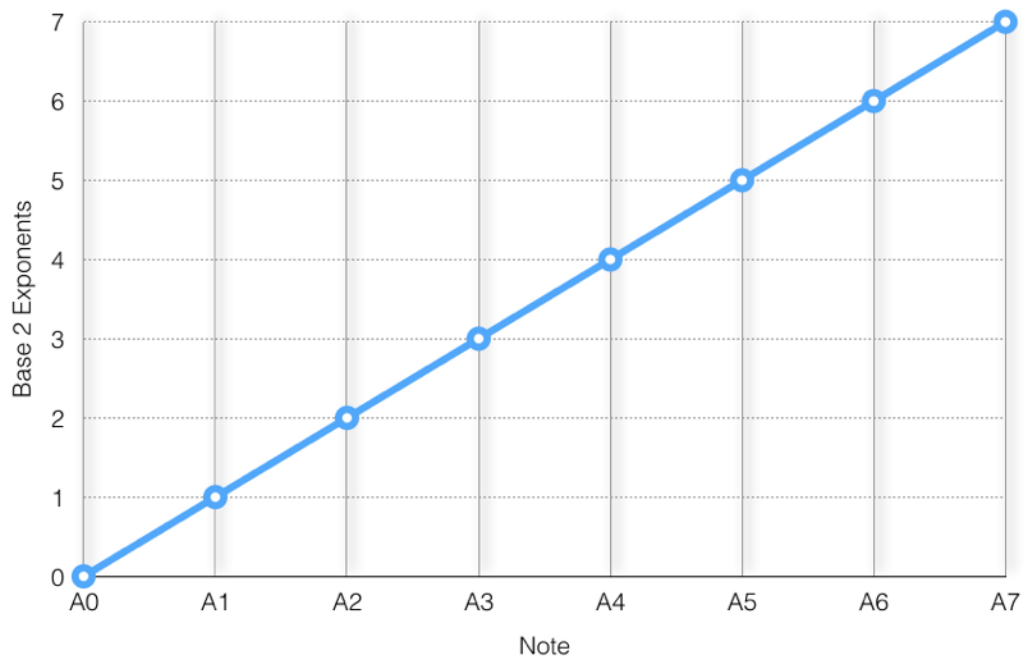

Figure 3-8. The linear relationship of base 2 exponents vs note, where Frequency = 2^n x 27.5

Now, when we take the difference of the octave numbers between any two neighbouring octave A's, the answer is "1". In other words, in our system here, each octave size has the value of 1.

But, if we would like each semitone to arbitrarily have the size of 100, that means the octave must measure 1200.

A simple way to do that would be to multiple the octave number by 1200.

Let's look at our formula for calculating the octave number,

$$n = \log_2(F_n) - 4.781359714 \quad (7)$$

If we want the answer to be in cents, we just multiple n by 1200.

Let's use the letter, c, for cents, so that,

$$c = 1200 \times n \quad (7.1)$$

Substitute the formula for n into (7.1) above.

$$c = 1200 \times [\log_2(F_n) - 4.781359714] \quad (8)$$

Multiply out each term by 1200 to get,

$$c = 1200\log_2(F_n) - 1200(4.781359714) \quad (8.1)$$

$$c = 1200\log_2(F_n) - 5737.6316562 \quad (9)$$

It is very important that we take some time to identify the assumptions we have made.

Cents is a relative measurement. It measures the difference between two frequencies. It does not give an absolute value of a frequency, unless we have arbitrarily choosen one frequency to be 0 cents.

In our systrem created here, we have done that. By linking octave number loosely with cents (which was only done in this article to help us develop the cents formula) we have arbitrarily set A0 = 0 cents.

So, by using (9), we can calculate the cents difference between any note and A0.

Example: How many cents is A2 above A0.

Solution, use (9)

$$c = 1200\log_2(F_n) - 5737.6316562 \qquad (9)$$

where $F_n = 110$ from table 3.5.

$$c = 1200\log_2(110) - 5737.6316562 \qquad (9.1)$$

Compute $1200\log_2(110)$

I do not have a $\log_2$ (log to the base 2) function on my calculator, but I do have $\log_{10}$.

Here we have to use a special property of logs that we will use again later.

We can use this property:

Given the log to the base a of b,

$$\log_a b$$

we can calculate it using,

$$\log_x b \,/\, \log_x a$$

Think: To calculate log to an unknown base of something, use log to a known base of the old something, divided by log to the known base of the old unknown base.

With this formula, we can calculate a $\log_a$ (log to the base a) expression, where we do not have a $\log_a$ function, using a log function we do have, $\log_x$.

In our specific case, we can calculate a $\log_2$ (log to the base 2) expression, where we do not have a $\log_2$ function, using a log function we do have, $\log_{10}$.

So,

$$1200\log_2(110)$$
$$= 1200 \times [\log_{10}(110) / \log_{10}(2)]$$
$$= 1200 \times [2.0413927 / 0.3010300]$$
$$= 1200 \times 6.7813597$$
$$= 8137.6316562$$

Subbing into (9.1)

$$c = 1200\log_2(110) - 5737.6316562 \qquad (9.1)$$
$$c = 8137.6316562 - 5737.6316562$$
$$c = 2400$$

2400 cents is 2 octaves and A2 is two octaves above A0.

CALCULATING CENTS BETWEEN ANY TWO A's

So now, if we want to calculate the difference in cents between any two A's (let's call that captital C) we just calculate the cents of each A, and subtract.

$$C = c_2 - c_1 \qquad\qquad (10)$$

where c_2 is the absolute cents of the higher octave A, and c_1 is the absolute cents of the lower octave A..

(It is worth repeating that cents is a relative value. In our case, we've chosen A0 to have an octave number of 0, and hence, an absolute cents value of 0, but we could have chosen any note to be zero. Cents, really, is only used to give a consistent value to a difference of *two* frequencies, and is not useful to give an absolute value of only one frequency.)

So, let's substitute c_1 and c_2 into the absolute cents formula (9).

First, we start with equation (9),

$$c = 1200\log_2(F_n) - 5737.6316562 \qquad (9)$$

and substitute c_1 and c_2 to get,

$$c_1 = 1200\log_2(F_{n1}) - 5737.6316562 \qquad (11)$$
$$c_2 = 1200\log_2(F_{n2}) - 5737.6316562 \qquad (12)$$

Substitute these expressions for c1 and c2 into (10),

$$C = c_2 - c_1 \qquad\qquad (10)$$

and we get,

$$C = 1200\log_2(F_{n2}) - 5737.6316562 - [1200\log_2(F_{n1}) - 5737.6316562]$$

Expanding, we get,

$$C = 1200\log_2(F_{n2}) - 5737.6316562 - 1200\log_2(F_{n1}) + 5737.6316562$$

The 5737.6316562's cancel out, to give,

$$C = 1200\log_2(F_{n2}) - 1200\log_2(F_{n1})$$

Take out the 1200 multiple, to get,

$$C = 1200[\log_2(F_{n2}) - \log_2(F_{n1})] \qquad (13)$$

Let's try it.

What is the cents difference between A2 and A6?

We can use (13),

$$C = 1200[\log_2(F_{n2}) - \log_2(F_{n1})] \qquad (13)$$

where n2 = 6 for A6, and n1 = 2 for A2.

Use table 3-5 to get the frequencies of A2 and A6,

$$F_6 = A6 = 1760$$

and,

$$F_2 = A2 = 110$$

Substituting n1, n2, F_{n1}, and F_{n2} into (13), and we get,

$$C = 1200[\log_2(1760) - \log_2(110)] \qquad\qquad (13)$$
$$C = 1200[10.78135971 - 6.78135971]$$
$$C = 1200[4]$$
$$C = 4800$$

Which is indeed four octaves! 4 x 1200 = 4800.

WHAT ABOUT INTERVALS SMALLER THAN AN OCTAVE?

Well, if we know the frequency of each note, then we can just input it into (13),

$$C = 1200[\log_2(F_{n2}) - \log_2(F_{n1})] \qquad\qquad (13)$$

The formula will work for intervals smaller than an octave.

But wait! Remember that nifty relationship I told you about above in (5.1) and (5.2)? Namely,

$$\log_x(a/b) = \log_x(a) - \log_x(b) \qquad\qquad (13.1)$$

Well, we can flip it around and get,

$$\log_x(a) - \log_x(b) = \log_x(a/b) \qquad\qquad (13.2)$$

The differences of logs to the same base can also be written as the log (same base) of the ratio of those two numbers.

Let's look at (13) again,

$$C = 1200[\log_2(F_{n2}) - \log_2(F_{n1})] \qquad\qquad (13)$$

See the difference of the logs to base 2 in there? We can that rewrite as,

$$C = 1200\log_2(F_{n2}/F_{n1}) \qquad\qquad (14)$$

One more step, before we get our final formula, is to notice that $\log_2$ operations are not common on most calculators, as already mentioned. In fact, the only common log functions are $\log_{10}$ and $\log_e$. [$\log_e$ is called the natural logarithm and is sometimes shown as just, ln, (the small letter L followed by n). Discussion of natural log is beyond the scope of this book.]

The only reason why I am making this step, is to explain to you why the formula looks the way it does when you do a google search or look in a book. It will not look like (14), even though (14) works if you can calculate a base 2 log.

Remember that, if you want to take $\log_2$ of a number, you won't find that function on a common calculator.

Also remember the nifty relationship with logs that we can use.

If you don't have the log of a base you need, you can use this formula to get the answer,

$$\log_a b = \log_x b \,/\, \log_x a \qquad\qquad (14.1)$$

where "a" is the base log function you do *not* have, and "x" is the base you *do* have. You can choose x = 10 because most calculators *do* have $\log_{10}$ functions.

So, if you have at least one log function on your calculator, you can use that, with a = 2, and x equal to the log's base of your calculator's log function.

For example, assume you only have the $\log_{10}$ on your calculator. If you need a $\log_2$ operation, just use (14.1),

$$\log_a b = \log_x b \,/\, \log_x a \qquad\qquad (14.1)$$

where a = 2 and x = 10.

which gives us,

$$\log_2 b = \log_{10} b \,/\, \log_{10} 2$$

So, using our cents formula (14) already determined,

$$C = 1200\log_2(F_{n2}/F_{n1}) \tag{14}$$

we can calculate our $\log_2$ function using $\log_{10}$, with (14.1),

$$\log_a b = \log_x b / \log_x a \tag{14.1}$$

where $a = 2$, and $b = F_{n2}/F_{n1}$, and $x = 10$, which becomes

$$\log_2(F_{n2}/F_{n1}) = \log_{10}(F_{n2}/F_{n1}) / \log_{10}2 \tag{15}$$

Substitute (15) into,

$$C = 1200\log_2(F_{n2}/F_{n1}) \tag{14}$$

and we get,

$$C = 1200 \log_{10}(F_{n2}/F_{n1}) / \log_{10}2 \tag{16}$$

where F_{n1} and F_{n2} are two frequencies, and we want to know how far apart they are, in cents.

I've also seen it written as

$$\mathbf{C = 1200 \ln(F_{n2} / F_{n1}) / \ln2} \tag{17}$$

because the natural log, ln, is also a common log function on calculators, but they both work.

So, (17) is one of the cents formulas, and how it's derived. I've looked it up before but never really understood why the ln function was used.

USEFUL RELATIONSHIPS TO REMEMBER

Just as an aside, sometimes we may like to know how many Hertz are in a cent. Well there's no one answer because for each semitone, the number of Hz difference changes, while the cents size of a semitone remains constant at 100 cents.

But, as a piano tuner, there is one Hertz/cents relationship you might want to memorize; the cents difference between 440Hz and 441Hz. That is because we often start tuning the piano at A4 and listen to the beats produced between A4 and the A440 tuning fork.

Let's use (17) with F_{n1} = 440Hz, and F_{n2} = 441Hz,

$$C = 1200 \ln(F_{n2}/F_{n1}) / \ln2 \qquad\qquad (17)$$
$$C = 1200 \ln(441/440) / \ln2$$
$$C = 3.930158439 \cong 4 \text{ cents}$$
($\cong$ means "is *approximately* equal to")

So, when you hear one beat per second at A4 when you're playing the A440 tuning fork against A4, you know you are about 4 cents off.

Also, at A5, the 1 Hz difference is 2 cents because the Hz double an octave up.

Check that with (17) and F_{n1} = 880Hz, and F_{n2} = 881Hz,

$$C = 1200 \ln(F_{n2}/F_{n1}) / \ln2$$
$$C = 1200 \ln(881/880) / \ln2$$
$$C = 1.966194475 \cong 2 \text{ cents}$$

Table 3-10 shows the approximate cents offset for a 1hz difference at each A on the piano.

A	Frequency(Hz)	1 Hz Difference in Cents	Rounded
A0	27.5	61.83	62
A1	55	31.19	32
A2	110	15.67	16
A3	220	7.85	8
A4	440	3.93	4
A5	880	1.97	2
A6	1760	0.98	1
A7	3520	0.49	0.5

Table 3-10. Cents Offset for 1Hz difference at Each Piano A.

The next few sections will deal with different commas.

(Next Section - 3.8 THE PYTHAGOREAN COMMA)